

Multilinear generating functions for Charlier polynomials

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Abstract

Charlier configurations provide a combinatorial model for Charlier polynomials. We use this model to give a combinatorial proof of a multilinear generating function for Charlier polynomials. As special cases of the multilinear generating function, we obtain the bilinear generating function for Charlier polynomials and formulas for derangements.

1 Introduction

Charlier polynomials have been studied using combinatorial methods in [5], [8], [10], [11], [15], and [16]. In this paper, we prove a multilinear generating function for Charlier polynomials using the combinatorial model of Charlier configurations [10, 11] and the approach of Foata and Garsia [6] in their proof of Slepian's multilinear extension of the Mehler formula for Hermite polynomials [12]. We then obtain some formulas for derangements as special cases of this generating function.

The Charlier polynomials are usually defined by the formula

$$c_n(a, r) = {}_2F_0(-n, -a; -; -r^{-1}) = \sum_{k=0}^n \binom{n}{k} (-a)_k \frac{r^{-k}}{k!},$$

where $(u)_k = u(u+1)\cdots(u+k-1)$. In order to assign convenient weights in the combinatorial model, we work with renormalized Charlier polynomials $C_n(a, r)$ defined by

$$C_n(a, r) = r^n c_n(-a, r) = \sum_{k=0}^n \binom{n}{k} (a)_k r^{n-k}.$$

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Our main result is the multilinear generating function

$$\sum_{(n_{ij})} \frac{\prod_{1 \leq i < j \leq k} x_{ij}^{n_{ij}}}{\prod_{1 \leq i < j \leq k} n_{ij}!} C_{n_1}(a_1, r_1) \cdots C_{n_k}(a_k, r_k) = \prod_{1 \leq i < j \leq k} e^{r_i r_j x_{ij}} \sum_{(n_{ij})} \prod_{1 \leq i \leq k} \frac{(a_i)_{n_i}}{(1 - \sum_{j \neq i} r_j x_{ij})^{n_i + a_i}} \frac{\prod_{1 \leq i < j \leq k} x_{ij}^{n_{ij}}}{\prod_{1 \leq i < j \leq k} n_{ij}!}, \quad (1)$$

where each sum runs over all $k \times k$ symmetric matrices (n_{ij}) with non-negative integral entries and with diagonal entries zero, $n_i = \sum_{j=1}^k n_{ij}$ for $1 \leq i \leq k$, and $x_{ij} = x_{ji}$.

To give a combinatorial proof of (1), we begin with a discussion of Charlier configurations and their representation by digraphs in section 2. Then in section 3 we give the combinatorial proof of the multilinear generating function. The main idea of the proof is to show that both sides of the formula count the same set of digraphs. We discuss the special cases of the multilinear generating function in section 4.

2 Charlier Configurations

Let $[n]$ denote the set $\{1, 2, \dots, n\}$.

Definition 2.1 A Charlier configuration on the set S is a pair $\Phi = ((A, \sigma), B)$, where (A, B) is an ordered partition of S and σ is a permutation of A .

A Charlier configuration is called a partial permutation in [10]. The configuration Φ can be represented by a digraph with vertex set S and with an edge from i to j if and only if $\sigma(i) = j$. Figure 1 shows a Charlier configuration on $[10]$.

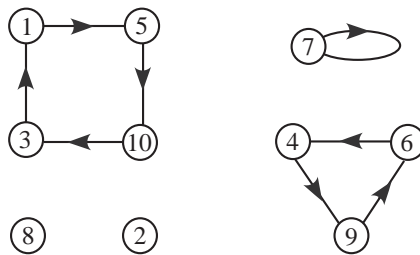


Figure 1: Charlier configuration on $[10]$

Here $A = \{1, 3, 4, 5, 6, 7, 9, 10\}$, $B = \{2, 8\}$, and $\sigma = (7) (4 \ 9 \ 6) (1 \ 5 \ 10 \ 3)$ in disjoint cycle notation.

2.1 Combinatorial interpretation of Charlier polynomials.

We assign a weight to a Charlier configuration Φ by assigning a weight a to each cycle of σ and a weight r to each point of B . Then the weight of Φ is $a^{\text{cyc}(\sigma)} r^{|B|}$ where $\text{cyc}(\sigma)$ denotes the number of cycles in σ . (If we had not renormalized the Charlier polynomials, we would assign the weight $-a$ to each cycle of σ and the weight $1/r$ to each point of A .) We use the following well-known facts about generating functions for permutations, which are proved, for example, in [13, p. 19].

Fact 1. $\sum_{k=0}^n c(n, k) a^k = (a)_n$ where $c(n, k)$ is the number of permutations of $[n]$ with exactly k cycles (the unsigned Stirling number of the first kind).

Fact 2. The exponential generating function for all permutations, with cycles weighted by a , is $(1 - z)^{-a}$.

Let \mathcal{C}_S denote the set of Charlier configurations on S . Then it follows easily from Fact 1 that $C_n(a, r)$ is sum of the weights of the elements of $\mathcal{C}_{[n]}$.

3 Combinatorial Proof of the Multilinear Formula

We assume that the reader is familiar with enumerative applications of exponential generating functions, as described, for example, in [14, Chapter 5] and [3]. The product formula and the exponential formula for exponential generating functions discussed in these references play an important role in the combinatorial proof of the multilinear formula. The theory of species (as used in [10] and [11]) could be used to provide a proof of the formula as well.

The formula (1) could be proved by interpreting it as a multivariable exponential generating function in the variables x_{ij} , which would require the use of digraphs with multiple sets of labels. The proof is simpler if we use exponential generating functions in only one variable, so that we can use a single set of labels. To accomplish this, we rewrite the formula by replacing x_{ij} with zx_{ij} . Now we can think of the formula as an exponential generating function in the single variable z . The formula is now

$$\sum_{(n_{ij})} \frac{\prod_{1 \leq i < j \leq k} x_{ij}^{n_{ij}}}{\prod_{1 \leq i < j \leq k} n_{ij}!} C_{n_1}(a_1, r_1) \cdots C_{n_k}(a_k, r_k) z^{\sum n_{ij}} =$$

$$\prod_{1 \leq i < j \leq k} e^{r_i r_j x_{ij} z} \prod_{1 \leq i \leq k} \frac{1}{(1 - z \sum_{j \neq i} r_j x_{ij})^{a_i}} \sum_{(n_{ij})} \prod_{1 \leq i \leq k} \frac{(a_i)_{n_i}}{(1 - z \sum_{j \neq i} r_j x_{ij})^{n_i}} \frac{\prod_{1 \leq i < j \leq k} x_{ij}^{n_{ij}}}{\prod_{1 \leq i < j \leq k} n_{ij}!} z^{\sum n_{ij}}. \quad (2)$$

We will prove this formula, which is equivalent to (1). We begin with a description of the digraphs counted by the left side of the formula.

3.1 Digraphs counted by the left side.

We can rewrite the left side of (2) as follows:

$$\sum_{n \geq 0} \frac{z^n}{n!} \sum_{\substack{(n_{ij}) \\ \sum_{i < j} n_{ij} = n}} \frac{n!}{\prod_{1 \leq i < j \leq k} n_{ij}!} \prod_{1 \leq i < j \leq k} x_{ij}^{n_{ij}} C_{n_1}(a_1, r_1) \cdots C_{n_k}(a_k, r_k).$$

Let $(N_{ij})_{1 \leq i < j \leq k}$ be an ordered partition of $[n]$ such that $|N_{ij}| = n_{ij}$. For $j > i$, let $N_{ji} = N_{ij}$. Let $N_i = \cup_{j \neq i} N_{ij}$. Then $N_i \cap N_j = N_{ij}$. Since $n_i = \sum_j n_{ij}$, it follows that $|N_i| = n_i$. Let H be the set of all ordered tuples $((N_{ij}), \Phi_1, \dots, \Phi_k)$ such that

1. (N_{ij}) is an ordered partition of $[n]$ with the above properties.
2. Each Φ_i is a Charlier configuration on N_i , i.e., $\Phi_i \in \mathcal{C}_{N_i}$.

Then each point of $[n]$ is in exactly two configurations. This follows from the fact that each point is in exactly one N_{ij} and $N_i \cap N_j = N_{ij}$. To the Charlier configuration $\Phi_i = ((A_i, \sigma_i), B_i)$ we assign the weight $a_i^{\text{cyc}(\sigma)} r_i^{|B|}$. We also assign an additional weight of x_{ij} to each point of N_{ij} . The weight of a tuple in H is defined to be the product of the weights of its constituent Charlier configurations and its points. Then it is easy to see that the left side of (2) is the exponential generating function for H with these weights.

We associate a digraph to a tuple $((N_{ij}), \Phi_1, \dots, \Phi_k)$ in H by superimposing the digraphs of these k Charlier configurations on $[n]$ in which each Φ_i is on n_i of these vertices. Figure 2 shows such a digraph for $k = 3$. The configurations Φ_1, Φ_2, Φ_3 are respectively represented by solid lines, dashed lines, and dotted lines. Each vertex is in exactly two configurations and this is indicated by the two different circles around each vertex.

The tuple and the configurations corresponding to Figure 2 are given by

$$N_{12} = \{5, 8, 9, 11, 14, 16, 17\}, N_{13} = \{1, 2, 6, 12, 15\}, \text{ and } N_{23} = \{3, 4, 7, 10, 13, 18\}.$$

$$\Phi_1 = ((A_1, \sigma_1), B_1), \text{ where}$$

$$A_1 = \{1, 8, 9, 11, 12, 15, 16\}, B_1 = \{2, 5, 6, 14, 17\},$$

$$\sigma_1 = (9 \ 16 \ 15) (1 \ 12 \ 11 \ 8).$$

$$\Phi_2 = ((A_2, \sigma_2), B_2), \text{ where}$$

$$A_2 = \{3, 4, 5, 7, 8, 10, 11, 17, 18\}, B_2 = \{9, 13, 14, 16\},$$

$$\sigma_2 = (5 \ 17 \ 8 \ 11) (3 \ 10 \ 7 \ 4 \ 18).$$

$$\Phi_3 = ((A_3, \sigma_3), B_3), \text{ where}$$

$$A_3 = \{1, 4, 6, 7, 10, 13, 18\}, B_3 = \{2, 3, 12, 15\},$$

$$\sigma_3 = (6 \ 13) (1 \ 10 \ 7 \ 18 \ 4).$$

We may identify H with the set of these digraphs, for which so that the left side of (2) is the exponential generating function. We now enumerate these digraphs in another way: We consider their connected components, which are of three types, and use the product formula for exponential generating functions to show that the right side of (2) is also a generating function for H .

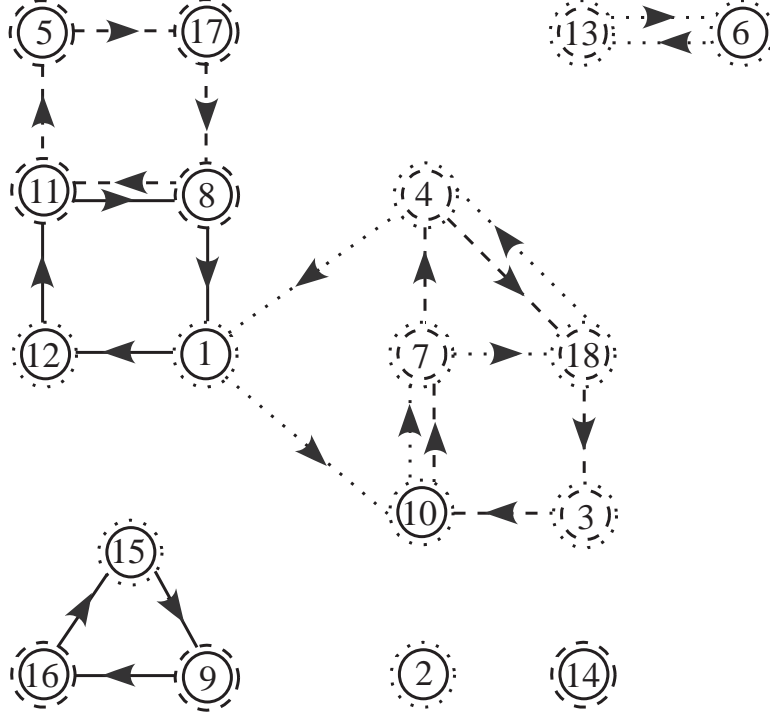


Figure 2: Digraph counted by left side for $k = 3$

3.2 Connected components of digraphs in H .

Let $((N_{ij}), \Phi_1, \dots, \Phi_k)$ be a tuple in H , where $\Phi_i = ((A_i, \sigma_i), B_i) \in C_{n_i}$, for $1 \leq i \leq k$. The connected components of the digraph representing this tuple are of the following three types:

For $i < j$, a **type 1_{ij} connected component** is an isolated vertex which is in Φ_i and Φ_j but not in σ_i or σ_j . In Figure 2, vertex 2 is of type 1_{13} and vertex 14 is of type 1_{12} . Such a vertex belongs to $B_i \cap B_j$ and is weighted by $r_i r_j x_{ij}$. It follows that the exponential generating function for digraphs all of whose components are of type 1_{ij} , which we call **type 1_{ij} digraphs**, is $e^{r_i r_j x_{ij} z}$.

A **type 2_i connected component** is a cycle of σ_i in which no vertex is in any other σ_j . In Figure 2, the cycle (6 13) is a type 2_3 component and (9 16 15) is a type 2_1 component. The cycle of a type 2_i component weighted by a_i and each vertex of the cycle is in some B_j and so is weighted by r_j and x_{ij} . A **type 2_i digraph** is a digraph in which every connected component is of type 2_i . Such digraphs can be considered as permutations in which each cycle is weighted by a_i and each vertex is weighted by some r_j and x_{ij} for some j . By a slight modification of Fact 2 in subsection 2.1, it follows that the exponential generating function for type 2_i digraphs is $(1 - z \sum_{j \neq i} r_j x_{ij})^{-a_i}$.

Any connected component that is not of type 1_i or 2_{ij} is called a **type 3 connected**

component. In a type 3 connected component, every vertex is in at least one permutation and every cycle contains at least one vertex that is also in another permutation.

The type 3 connected component from Figure 2 is shown in Figure 3. A **type 3**

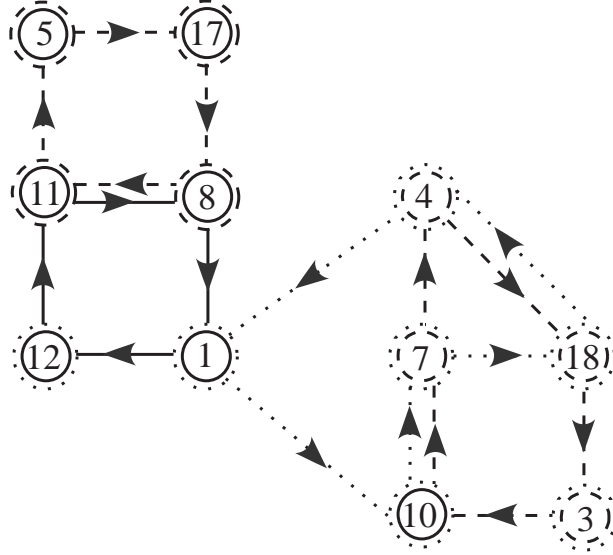


Figure 3: Type 3 connected component

digraph is a digraph all of whose connected components are of type 3. We say that a type 3 digraph is reduced if every vertex is in two permutations. Thus, a reduced type 3 digraph on n vertices is an ordered tuple $((N_{ij}), \Phi_1, \dots, \Phi_k)$ where each $\Phi_i = ((A_i, \sigma_i), \emptyset)$; i.e., each Φ_i is simply a permutation σ_i on n_i vertices. Figure 4 shows a reduced type 3 digraph on 7 vertices. Since each cycle of σ_i is weighted by a_i , the exponential generating

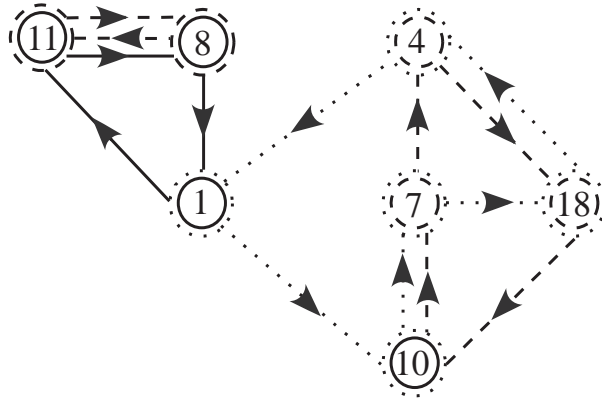


Figure 4: Reduced digraph on 7 vertices

function for reduced type 3 digraphs is

$$\sum_{n \geq 0} \frac{z^n}{n!} \sum_{\substack{(n_{ij}) \\ \sum_{i < j} n_{ij} = n}} \frac{n!}{\prod_{1 \leq i < j \leq k} n_{ij}!} \prod_{1 \leq i \leq k} (a_i)_{n_i} \prod_{1 \leq i < j \leq k} x_{ij}^{n_{ij}}.$$

Each vertex in a reduced type 3 digraph has two outgoing edges belonging to two different permutations. Any type 3 digraph can be obtained from a reduced type 3 digraph by replacing each outgoing edge at every vertex by a sequence of ordered edges. An outgoing edge in σ_i is replaced by a sequence of edges, such that each new vertex is in σ_i , but not in any other σ_j . Hence each new vertex is weighted by r_j and x_{ij} for some j . Thus the exponential generating function for type 3 digraphs is

$$\sum_{n \geq 0} \frac{z^n}{n!} \sum_{\substack{(n_{ij}) \\ \sum_{i < j} n_{ij} = n}} \frac{n!}{\prod_{1 \leq i < j \leq k} n_{ij}!} \prod_{1 \leq i \leq k} \frac{(a_i)_{n_i}}{(1 - z \sum_{j \neq i} r_j x_{ij})^{n_i}} \prod_{1 \leq i < j \leq k} x_{ij}^{n_{ij}}.$$

It follows from the product formula for exponential generating functions that the exponential generating function for digraphs in H is the product of the generating functions for all of the types of digraphs described above and this is

$$\begin{aligned} & \prod_{1 \leq i < j \leq k} e^{r_i r_j x_{ij} z} \prod_{1 \leq i \leq k} \frac{1}{(1 - z \sum_{j \neq i} r_j x_{ij})^{a_i}} \\ & \times \sum_{n \geq 0} \frac{z^n}{n!} \sum_{\substack{(n_{ij}) \\ \sum_{i < j} n_{ij} = n}} \frac{n!}{\prod_{1 \leq i < j \leq k} n_{ij}!} \prod_{1 \leq i \leq k} \frac{(a_i)_{n_i}}{(1 - z \sum_{j \neq i} r_j x_{ij})^{n_i}} \prod_{1 \leq i < j \leq k} x_{ij}^{n_{ij}}, \end{aligned}$$

which is equal to the right side of (2).

4 Specializations

For $k = 2$, the only parameter n_{ij} in (1) is n_{12} , and $n_1 = n_2 = n_{12}$. If we write n for n_{12} , a for a_1 , b for a_2 , r for r_1 , s for r_2 , and x for x_{12} then the multilinear formula (1) reduces to the bilinear formula

$$\sum_{n \geq 0} C_n(a, r) C_n(b, s) \frac{x^n}{n!} = e^{rsx} \sum_{n \geq 0} \frac{(a)_n (b)_n}{(1 - sx)^{n+a} (1 - rx)^{n+b}} \frac{x^n}{n!}. \quad (3)$$

Formula (2.47) in Askey's book [2] is equivalent to the case of (3) in which a and b are negative integers, and the general case is easily derived from this. Note that a and b are switched on the right side of the formula in the book.

Similarly, the case $k = 3$ of (1) may be written

$$\sum_{l,m,n} C_{l+m}(a,r)C_{l+n}(b,s)C_{m+n}(c,t)\frac{x^l y^m z^n}{l! m! n!} = e^{rsx+rtz+stz} \\ \times \sum_{l,m,n} \frac{(a)_{l+m}(b)_{l+n}(c)_{m+n}}{(1-sx-ty)^{l+m+a}(1-rx-tz)^{l+n+b}(1-ry-sz)^{m+n+c}} \frac{x^l y^m z^n}{l! m! n!}. \quad (4)$$

Some special cases of these formulas are worth mentioning. Setting $x = 0$ in (4) gives

$$\sum_{m,n} C_m(a,r)C_n(b,s)C_{m+n}(c,t)\frac{y^m z^n}{m! n!} \\ = e^{(ry+sz)t} \sum_{m,n} \frac{(a)_m(b)_n(c)_{m+n}}{(1-ty)^{m+a}(1-tz)^{n+b}(1-ry-sz)^{m+n+c}} \frac{y^m z^n}{m! n!}. \quad (5)$$

Formula (5) may be viewed as a Charlier polynomial analogue of a formula of Carlitz [4] for Hermite polynomials, which is a special cases of Slepian's multilinear extension of the Mehler formula [12].

By applying the fact that $C_n(0, 1) = 1$, we can find other simplifications. Thus setting $b = 0$ and $s = 1$ in (3) gives the usual exponential generating function for Charlier polynomials,

$$\sum_{n \geq 0} C_n(a, r) \frac{x^n}{n!} = e^{rx}(1-x)^{-a}.$$

Setting $c = 0$ and $t = 1$ in (4) gives a generalization of (3):

$$\sum_{l,m,n} C_{l+m}(a,r)C_{l+n}(b,s)\frac{x^l y^m z^n}{l! m! n!} = e^{rsx+ry+sz} \sum_l \frac{(a)_l(b)_l}{(1-sx-y)^{l+a}(1-rx-z)^{l+b}} \frac{x^l}{l!}. \quad (6)$$

4.1 Permutations.

The Charlier polynomials can be normalized in another way so as to count permutations by cycles and fixed points. We define polynomials $D_n(\alpha, u)$ by

$$D_n(\alpha, u) = \sum_{\pi \in \mathfrak{S}_n} \alpha^{\text{cyc}_{>1}(\pi)} u^{\text{fix}(\pi)},$$

where \mathfrak{S}_n is the set of permutations of $[n]$, $\text{cyc}_{>1}(\pi)$ is the number of cycles of π of length greater than 1, and $\text{fix}(\pi)$ is the number of fixed points of π .

We can express the polynomials $D_n(\alpha, u)$ in terms of Charlier polynomials. To a Charlier configuration $\Phi = ((A, \sigma), B)$ on $[n]$ we may associate the permutation π of $[n]$ such that $\pi(i) = \sigma(i)$ for $i \in A$ and $\pi(i) = i$ for $i \in B$. Conversely, given a permutation π of $[n]$, the corresponding Charlier configurations may be constructed by choosing an

arbitrary subset B of the set of fixed points of π and taking σ to be the restriction of π to $A = [n] \setminus B$. This construction yields the relation

$$C_n(a, r) = D_n(a, a + r)$$

and thus

$$D_n(\alpha, u) = C_n(\alpha, u - \alpha).$$

So formulas (3)–(6) may be rewritten as generating functions for the polynomials $D_n(\alpha, u)$. Of particular interest are the specializations $D_n(\alpha) = D_n(\alpha, 0) = C_n(\alpha, -\alpha)$, which count derangements (permutations without fixed points) by cycles, and $D_n = D_n(1) = C_n(1, -1)$, the number of derangements of $[n]$.

Making the appropriate substitutions in (3) gives

$$\sum_{n=0}^{\infty} D_n(\alpha) D_n(\beta) \frac{x^n}{n!} = e^{\alpha\beta x} \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(1 + \beta x)^{n+\alpha} (1 + \alpha x)^{n+\beta}} \frac{x^n}{n!}. \quad (7)$$

Formula (7) was proved by Gessel [7] as a special case of a generating function for $3 \times n$ Latin rectangles, using an approach similar to that of this paper. (See also [1] and [17].)

The corresponding specialization of (5) is

$$\begin{aligned} \sum_{m,n=0}^{\infty} D_m(\alpha) D_n(\beta) D_{m+n}(\gamma) \frac{y^m}{m!} \frac{z^n}{n!} \\ = e^{(\alpha y + \beta z)\gamma} \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\beta)_n (\gamma)_{m+n}}{(1 + \gamma y)^{m+\alpha} (1 + \gamma z)^{n+\beta} (1 + \alpha y + \beta z)^{m+n+\gamma}} \frac{y^m}{m!} \frac{z^n}{n!} \end{aligned}$$

and of (6) is

$$\sum_{l,m,n} D_{l+m}(\alpha) D_{l+n}(\beta) \frac{x^l}{l!} \frac{y^m}{m!} \frac{z^n}{n!} = e^{\alpha\beta x - \alpha y - \beta z} \sum_l \frac{(\alpha)_l (\beta)_l}{(1 + \beta x - y)^{l+\alpha} (1 + \alpha x - z)^{l+\beta}} \frac{x^l}{l!}.$$

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